We considered the problem of reconstructing the heat-flux density on one of the boundaries of an unbounded plate in which the process of heat transfer is described by a linear homogeneous heat-conduction equation. The boundary condition of the second kind at the other boundary of the plate was known. As the input data, we used the variation of temperature as a function of time at an interior point of the plate. The solution of the initial heat-conduction problem was represented in integral form. Furthermore, using the method of [1], we passed to a system of linear algebraic equations, for which we constructed the above-described algorithms. The inverse problem was solved both for the exact input data and for input data perturbed by means of a random-number device. When the input-temperature perturbations were up to 10% of the maximum value, a halt by the discrepancy principle was obtained within three to eight iterations, depending on the variant involved. When a constant descent parameter was used in analogous simulated examples, 30 to 60 iterations were required.

NOTATION

A, B, L, linear operators; u, element of the solution space U; $\bar{f}$, exact initial data; $\tilde{f}$, error in the initial data; δ, value of the error in the initial data; A^{-1}, inverse operator; $u^{(k)}(\tau)$, the k-th derivative of the function u; $\varphi_i(\tau)$, polynomials of degree $i-1$; A*, B*, L*, operators conjugate to the operators A, B, L; J(g), discrepancy functional; J'g, gradient of the discrepancy functional; β_n^i, depth of descent with respect to the i-th component of the antigradient of the discrepancy in the n-th iteration; τ_m, length of the observation interval.

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REGULARIZING ALGORITHM FOR INVERTING THE ABEL EQUATION

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The article presents a regularizing algorithm for solving the Abel equation using information on the statistics of the error of measurement of the right-hand side of the equation.

Optical methods have found widespread application in the diagnostics of electric arcs, impulse discharges, gas and plasma streams. The characteristics measured in the course of these operations are correlated with the sought local parameters of the object by the Abel equation [1]:

$$2\int_x^R \frac{\varphi(r)\,rdr}{(r^2-x^2)^{1/2}} = f(x), \quad x\in[0,\ R]. \tag{1}$$

Formally, the solution of $\varphi(r)$ can be determined by inverting the Abel equation, i.e.,

$$\varphi(r) = -\frac{1}{\pi}\int_r^R \frac{f'(x)\,dx}{(x^2-r^2)^{1/2}}, \quad r\in[0,\ R], \tag{2}$$

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where $f'(x)$ is the first derivative of the right-hand side of (1). The problem of calculating (2), like the problem of solving the initial equation (1), belongs to the incorrectly stated problems [2] whose characteristic trait is the instability of the solution as regards errors in the initial data. The instability of (2) is due to the unboundedness of the differentiation operator. Moreover, the inevitable errors of measurement of the right-hand side may infringe the property of continuous differentiability of the function f(x), and this leads to the absence of a solution. A number of works (see, e.g., the review article [3]) suggested two methods to overcome the above difficulties. The first method involves approximation of the function f(x) by a linear combination of polynomials that are orthogonal in the entire interval of determination of f(x); the second method involves the approximation of the values of the argument of the function f(x) in each separate interval by a polynomial of third to fifth degree. Though these methods ensure the existence of a continuous derivative, they have a definite shortcoming in the lack of constructive algorithms for selecting the parameters determining the degree of smoothing of the right-hand side of f(x) in dependence on the level of uncertainty of the initial data.

The present article suggests a regularizing algorithm for inverting the Abel equation on the basis of smoothing splines. It presents a procedure for selecting the parameter of the smoothing spline minimizing the rms error of smoothing the right-hand side of f(x).

Algorithm for Inverting the Abel Equation. In the experiment, the values of the right-hand side of f(x) are recorded at the points $x_i: 0 = x_1 < x_2 < \ldots < x_n = R$ with some additive error ξ_i, i.e., the initial data for solving Eq. (1) are represented by an n-dimensional vector with the coordinates $\tilde{f}_i = f(x_i) + \xi_i,\ i = \overline{1, n}$. As regards the uncertainty of measurement, we assume that the vector $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ has zero mathematical expectation and the correlation matrix $V_\xi = M[\xi\xi^T]$, where $M[\cdot]$ is the operator of mathematical expectation; T is the symbol of transposition.

To calculate $\varphi(r)$ on the basis of (2) it is necessary to determine some differentiated function approximating the right-hand side of f(x). We will examine this problem in the statement leading to cubic splines: it is essential to construct a function $S_{n,\alpha}(x)$, differentiated in the interval [0, R], which minimizes the functional [4]:

$$\Phi_\alpha[S(x)] = \alpha \int_0^R (S''(x))^2\, dx + \sum_{i=1}^{n} p_i^{-1} (\tilde{f}_i - S(x_i))^2,$$

where $p_i (p_i > 0)$ are the weight factors characterizing the significance of the i-th measurement. It was shown in [4] that the solution of this problem exists in the class of cubic splines, i.e., $S_{n,\alpha}(x)$ satisfies the conditions:

a) in each of the intervals $[x_i, x_{i+1}],\ i = \overline{1, n-1}$, the function $S_{n,\alpha}(x)$ is a cubic polynomial

$$S_{n,\alpha}(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3;$$

b) the function $S_{n,\alpha}(x)$ and its first two derivatives $S'_{n,\alpha}(x)$, $S''_{n,\alpha}(x)$ are everywhere continuous in [0, R], i.e., $S_{n,\alpha}(x) \in C^2_{[0,R]}$;

c) $S'_{n,\alpha}(0) = f'(0)$, $S'_{n,\alpha}(R) = f'(R)$ are the boundary conditions of the spline.

Remark. For the solution $\varphi(r)$ on the segment [0, R] to be bounded, the function f(x) has to satisfy the constraints $f'(0) = 0$, $f(R) = 0$. It is therefore necessary that $S'_{n,\alpha}(0) = 0$, and for determinacy we put $S'_{n,\alpha}(R) = 0$.

After we have calculated (with specified smoothing parameter) the coefficients a_i, b_i, c_i, d_i (see [4, 5]), the derivative $f'(x)$ for $x \in [x_i, x_{i+1}]$ is determined by the polynomial $S'_{n,\alpha}(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$. This makes it possible to reduce the inversion of (2) to the summing of the integrals of the type

$$\int_{x_i}^{x_{i+1}} (x - x_i)^k (x^2 - r^2)^{-1/2}\, dx, \quad k = \overline{0, 2}.$$

The values of these integrals are expressed by elementary functions, and this makes it possible to represent the regularized solution $\varphi_\alpha(r)$ of Eq. (1) in the form

$$\varphi_\alpha(r) = -\frac{1}{\pi}\left[I(r, r, x_m) + \sum_{i=m}^{n-1} I(r, x_i, x_{i+1})\right], \tag{3}$$

where

$$I(r, x_i, x_{i+1}) = (2c_i - 6d_i x_i)(D_{i+1} - D_i) + (b_i - 2c_i x_i +$$
$$+ 3d_i x_i^2 + 1.5 d_i r^2) \ln\left(\frac{x_{i+1} + D_{i+1}}{x_i + D_i}\right) + 1.5 d_i (x_{i+1} D_{i+1} - x_i D_i);$$
$$D_i = (x_i^2 - r^2)^{1/2},$$

and m is determined from the condition $x_{m-1} \leqslant r < x_m$.

Algorithm for Selecting the Smoothing Parameter. The selection of the smoothing parameter, which was previously considered specified, represents the chief difficulty in constructing the smoothing spline. The point is that when α is small, the smoothing of the uncertainty of measurement is slight, and in the derivative $S'_{n,\alpha}(x)$ oscillations appear which do not exist in $f'(x)$. When the values of α are large, the function $S_{n,\alpha}(x)$ is excessively smooth. Moreover, α has to be selected in such a way that when the level of the uncertainty of measurement tends to zero, the smoothing spline converges to the interpolating spline $S_n(x)$, which, in addition to a), b), c), satisfies the condition d) $S_n(x_i) = f(x_i)$, $i = \overline{1, n}$.

Voskoboinikov [5] suggested a criterion of optimum approximation to the experimental information, and on its basis an algorithm for estimating the optimum value of the parameter α was devised, minimizing the rms error of smoothing. As to its computing procedure, the algorithm is analogous to the algorithm for selecting the regularization parameter which was explained in [6]. The present work presents only the basic relationships for calculating the smoothing parameter.

We designate by $e(\alpha)$ the discrepancy vector with the coordinates $e_i(\alpha) = \tilde{f}_i - S_{n,\alpha}(x_i)$. A sufficient condition of optimality of the value α is the identity [5]:

$$V_e(\alpha) = \alpha V_\xi H^{\mathrm{T}}(\alpha H P H^{\mathrm{T}} + A)^{-1} H P, \tag{4}$$

where $V_e(\alpha) = M[e(\alpha)e^{\mathrm{T}}(\alpha)]$ is the matrix of the second moments of the vector $e(\alpha)$. The matrices H, P, A were determined in [5]. To calculate the value of the smoothing parameter α_g that does not conflict in the statistical sense with the identity (4), we devise a calculating procedure with respect to $\gamma = 1/\alpha$ in the form [5]:

$$\gamma_{k+1} = \gamma_k - \frac{[R(\gamma_k) - n]}{R'(\gamma_k)}, \quad k = 0,\ 1,\ 2,\ \ldots,\ \gamma_0 > 0, \tag{5}$$

where

$$R(\gamma) = \tilde{f}^{\mathrm{T}} V_\xi^{-1} P H^{\mathrm{T}} (H P H^{\mathrm{T}} + \gamma A)^{-1} H \tilde{f}.$$

As α_g we adopt the value $1/\gamma_k$ at which $R(\gamma_k) \in [\vartheta_n(\beta/2),\ \vartheta_n(1-\beta/2)]$, where $\vartheta_n(\beta/2) - \beta/2$ is the quantile of the χ^2-distribution with n degrees of freedom; β is the magnitude of the error of first kind in verifying the hypothesis (4).

Is the Algorithm of Inverting (3) Regularizing? Inversion of the Abel equation with exact right-hand side can be represented in operator form $\varphi(r) = B(r, x)Df(x)$, $B(r, x)$ is the integral operator with kernel $(x^2 - r^2)^{-1/2}$, and D is the differentiation operator. The constructed inversion (3) can also be written in the form $\varphi_\alpha(r) = B(r, x) D T_{n,\alpha}(x)\tilde{f}$, where $T_{n,\alpha}(x)$: $E_n \to C^2_{[0,R]}$ is the operator putting into correspondence the vector $\tilde{f} \in E_n$ and the twice-differentiated function $S_{n,\alpha}(x) \in C^2_{[0,R]}$.

For the algorithm of inverting (3) to be regularizing, it suffices to prove the convergence [2]:

$$\lim_{n\to\infty,\,\delta\to 0}\ \max_{r\in[0,R]} |\varphi(r) - \varphi_\alpha(r)| = 0, \tag{6}$$

where $\delta = \mathrm{Sp}[V_\xi]$ is the trace of the matrix V_ξ. The operator $B(r, x)$ is perfectly continuous, therefore (6) converges if

$$\lim_{n\to\infty,\,\delta\to 0}\ \max_{x\in[0,R]} |f(x) - S_{n,\alpha}(x)| = 0, \tag{7}$$

We require two statements.

Statement 1 [5]. The algorithm (5) for selecting the smoothing parameter guarantees the convergence

$$\lim_{\delta\to 0}\ \max_{x\in[0,R]} |S_n(x) - S_{n,\alpha}(x)| = 0.$$

Statement 2 [4, p. 90]. If $f(x) \in C^1_{[0,R]}$ and the integration spline $S_n(x)$ satisfies the boundary conditions c), then

$$\lim_{\Delta_n\to 0}\ \max_{x\in[0,R]} |f(x) - S_n(x)| = 0,$$

where $\Delta_n = \max|x_{i+1} - x_i|$. From these two statements and from the inequality

$$|f(x) - S_{n,\alpha}(x)| \leqslant |f(x) - S_n(x)| + |S_n(x) - S_{n,\alpha}(x)|$$

the convergence of (7) follows directly. Thereby we have proved the following statement.

Statement 3. The algorithm (3) for inverting the Abel equation with the parameter α_g, determined by procedure (5), is regularizing in the space $C_{[0,R]}$. It is realized in the form of a subprogram in the computer language FORTRAN IV and is widely used at the Institute of Thermophysics, Siberian Branch of the Academy of Sciences of the USSR, in processing experimental results.

In conclusion we want to dwell on some computing features of the suggested algorithm.

1. The algorithm makes it possible to construct the solution of the Abel equation with correlated uncertainties of measurement of the right-hand side (the structure of the matrix is different from the diagonal structure).

2. The algorithm is rapid. For instance, the computer time for constructing $\varphi_\alpha(r)$ with n = 40 was about 0.8 sec (the calculation was carried out on an M4030 computer), which is 2 to 3 orders of magnitude less than with other regular methods of solving Eq. (1).

3. The algorithm is economical as regards utilization of the internal memory of the computer. For instance, for storing the initial information, the working bodies of information, and the solution, the required storage capacity is $10 \times n$ words; this makes it possible to construct a solution vector with the dimension of several hundreds or even thousands of points without having recourse to the external memory of the computer.

NOTATION

$\varphi(r)$, $f(x)$, solution and right-hand side of the Abel equation, respectively; $\tilde{f}_i$, value of the right-hand side measured at point x_i; ξ_i, uncertainty of the i-th measurement; n, number of measurements of the right-hand side; V_ξ, correlation matrix of the uncertainty of measurement; α, smoothing parameter; $S_n(x)$, interpolating spline; $S_{n,\alpha}(x)$, smoothing spline; a_i, b_i, c_i, d_i, coefficients of the smoothing spline; $\varphi_\alpha(r)$, regularized solution of the Abel equation; $e(\alpha)$, discrepancy vector; $Sp[V_\xi]$, trace of the matrix V_ξ.

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